Study and implementation of an algebraic method to solve systems with fuzzy coefficients

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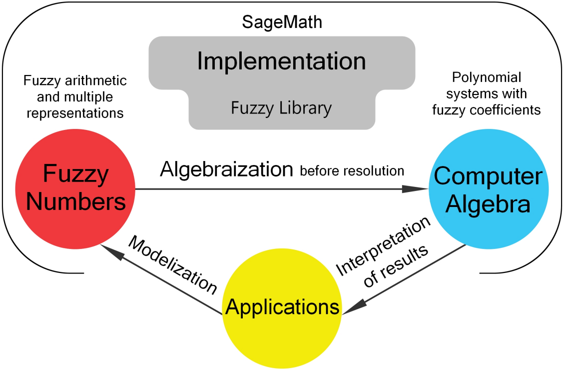
Context : Study and implementation of an algebraic method to solve systems with fuzzy coefficients



Approach

Recent approach of a global method based on computer algebra, an algebraic technique producing an exact result : the algorithm of Wu Wen Tsun.

M. Boroujeni, A. Basiri, S. Rahmany, and A. Valibouze. Finding solutions of fuzzy polynomial equations systems by an algebraic method. Journal of Intelligent Fuzzy Systems, 2016.

Resolution of polynomial systems with fuzzy coefficients

Compute the real solutions of the system :

$$AX + B = CX + D,$$

A,B,C,D matrices with fuzzy coefficients, X vector of real variables
Coefficients = triangular fuzzy numbers

$$\sum_{i=1}^{n} \widetilde{a}_{li} \cdot x_i + \widetilde{b}_l = \sum_{i=1}^{n} \widetilde{c}_{li} \cdot x_i + \widetilde{d}_l$$

for $0 \leq l \leq s$.

① Passage to the parametric system : twice as many equations, one parameter r, then an intermediate system S : the collected crisp system

② Computation of characteristic sets of S by Wu Wen Tsun's triangular decomposition algorithm

③ Correspondences between the quasi-varieties of these sets and the positive solutions of the system of fuzzy polynomials : find the exact solutions

$$V(F) = \bigcup_{C \in Z} V(C/I_C) \quad \text{avec } I_C = \prod_{p \in C} initial(p)$$

Summary

- Theory of Fuzzy Numbers
- Algebraic resolution (Wu's method)
- Passage from fuzzy to algebraic
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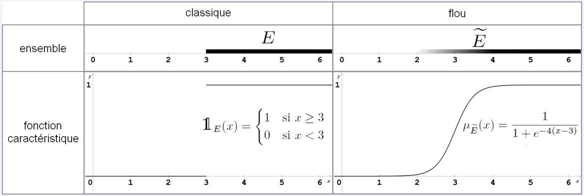
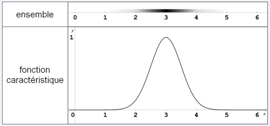
Fuzzy Numbers

➤ theory developed by Lotfi Zadeh in 1965

- Fuzzy sets : the membership function represents a degree of validity



- Advantages provided by fuzzy numbers : capturing uncertainty around a given value



$\mu_{\widetilde{n}}(x)$ represents the degree of validity of the proposition "x is the value of $\widetilde{n}$"

Principle of Fuzzification

From a function f of the form

$$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x_1, \ldots, x_n) & \longmapsto & y = f(x_1, \ldots, x_n) \end{array}$$

we induce the following function $\widetilde{f}$

$$\begin{array}{rccc} \widetilde{f}: & \mathfrak{B}(\mathbb{R})^n & \longrightarrow & \mathfrak{B}(\mathbb{R}) \\ & (\widetilde{x}_1, \ldots, \widetilde{x}_n) & \longmapsto & \widetilde{y} = \widetilde{f}(\widetilde{x}_1, \ldots, \widetilde{x}_n) \end{array}$$

where $\mathfrak{B}(\mathbb{R})$ is the class of fuzzy numbers of $\mathbb{R}$.

- The function f acts on real numbers, mean values.
- The interest of the function $\widetilde{f}$ is to keep the coherence of the action of the function f on fuzzy numbers, more complex, taking into account their mean value, their support and the general form of their membership function.

Principle of Fuzzification

This principle lays the foundation for fuzzy arithmetic. Fix $\widetilde{m}$ and $\widetilde{n}$ two fuzzy numbers.

The sum :

$$\mu_{\widetilde{m \oplus n}}(z) = \max_{z=x+y} \min(\mu_{\widetilde{m}}(x), \mu_{\widetilde{n}}(y))$$

$(x, y, z) \in \mathbb{R}^3$.
The law $\oplus$ is associative and commutative.

The opposite :

$$\mu_{\widetilde{-m}}(z) = \max_{z=-x} \min(\mu_{\widetilde{m}}(x)) = \mu_{\widetilde{m}}(-z)$$

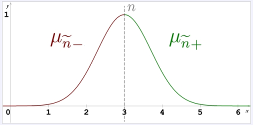
This is the symmetric function of $\mu_{\widetilde{m}}$ with respect to the y-axis

- For a fuzzy number $\widetilde{m}$ whose support is not reduced to its mode, $\widetilde{m} \oplus \widetilde{-m} \neq 0$, because $\widetilde{m}$ has no symmetric element for the law $\oplus$.

Tuple representation for finite supports

- The tuple representation proposed by Dubois and Prade in 1977

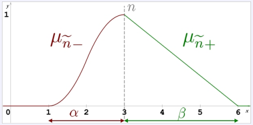
Infinite support :



mean value n

restrictions types : gaussians

Finite support :

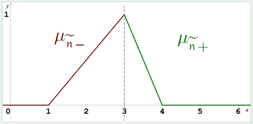


triplet $(n,\ \alpha,\ \beta)$

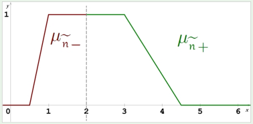
restrictions types : quadratic and linear

Simple families

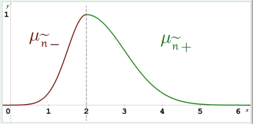
- types of restrictions $\mu_{\tilde{n}-}$ and $\mu_{\tilde{n}+}$ induce families of fuzzy numbers.



Triangular



Trapezoïdal



Gaussian

- The computations are carried out within the same family, two distinct simple families are incompatible with each other.

A family is defined by a unique couple of functions (L, R)

Let L and R defined from $[0,+\infty[$ to $[0,1]$ with $L(0)=R(0)=1$, $L(1)=R(1)=0$, continue and decreasing on their domain.

Let $\widetilde{m}=(m,\alpha,\beta)$ and $\widetilde{n}=(n,\gamma,\delta)\in\mathfrak{F}(L,R)$, the family L-R, so :

$$\mu_{\widetilde{m}-}(x)=L\left(\frac{m-x}{\alpha}\right),\quad \mu_{\widetilde{m}+}(x)=R\left(\frac{x-m}{\beta}\right),$$

$$\mu_{\widetilde{n}-}(x)=L\left(\frac{n-x}{\gamma}\right),\quad \mu_{\widetilde{n}+}(x)=R\left(\frac{x-n}{\delta}\right).$$

Arithmetic on tuples

The sum is a fuzzy number $L-R$:

$$\widetilde{m}\oplus\widetilde{n}=(m,\alpha,\beta)\oplus(n,\gamma,\delta)=(m+n,\alpha+\gamma,\beta+\delta)$$

The opposite is a fuzzy number $R-L$:

$$\widetilde{-m}=-(m,\ \alpha\ ,\ \beta)=(-m,\ \beta\ ,\ \alpha)$$

- The equations are independent of the analytical expressions of L and R : the operations are performed on the triplets without neither L nor R being known a priori.

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Tools

Let $\mathfrak{R} = \mathbb{K}[x_1, x_2, \ldots, x_n]$, $\mathbb{K}$ field of characteristic zero, with the lexicographic order. Let $p, q \in \mathfrak{R}$ such that $q \notin \mathbb{K}$.

- class$(p) = \max\{\, i \in \{1, \ldots, n\} \mid x_i$ appears in $p\,\}$.
  The leading coefficient of p in $x_{class(p)}$ is denoted init(p).
- p is reduced with respect to q if and only if $deg_{x_c}(p) < deg_{x_c}(q)$ where $c = class(q) \neq 0$.
- An ordered set $F = \{f_1, \ldots, f_r\}$ is called a triangular set if $r = 1$ or if $class(f_1) < \cdots < class(f_r)$. It is called an ascending set if each f_j is reduced with respect to each f_i, for $i < j$.

Pseudo-division

Let $f, g \in \mathfrak{R}$ et $c = class(f)$. So there is an equation of the form

$$init(f)^m g = qf + prem(g, f)$$

with $q \in \mathfrak{R}$ the pseudo-quotient, $prem(g, f) \in \mathfrak{R}$ the pseudo-remainder, $m \geq 0$ and $r = 0$ or r is reduced with respect to f.
For a finite subset $G \subset \mathfrak{R}$, we set

$$prem(G, F) = \{prem(g, F) \mid g \in G\}.$$

Characteristic set

A ascending set B in $\mathfrak{R}$ is called characteristic set of $F \subset \mathfrak{R}$ if $B \subset < F >$ and $prem(F, B) = \{0\}$.

Quasi-algebraic variety

The set

$$V(F) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n \,|\, f(a_1, \ldots, a_n) = 0, \forall f \in F\}$$

is the variety defined by F.

For $G \subset \mathfrak{R}$, $V(F/G) = V(F) \backslash V(G)$ is a quasi-algebraic variety.

Wu Principle

Let B be a characteristic set of $F \subset \mathfrak{R}$. So

$$V(F) = V(B/I_B) \bigcup \cup_{b \in B} V(F \cup B \cup \{init(b)\})$$

where $I_B = \prod_{b \in B} init(b)$.

- By repeating Wu's Principle Theorem, for each $F \cup B \cup \{init(b)\}$, $b \in B$, the procedure will end in a finite number of steps.
- The Wu algorithm allows to express the variety $V(F)$ as a finite union of quasi-algebraic varieties of characteristic sets $V(B/I_B)$. Finding $V(F)$ becomes easy because these caracteristic sets are easy to solve.

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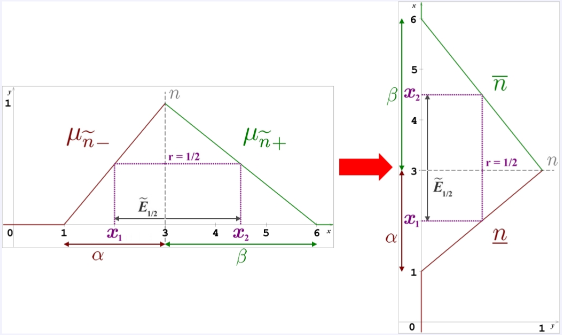
Parametric representation

The parametric form of a fuzzy number $\widetilde{n}$ is an ordered pair $[\underline{n}(r),\ \overline{n}(r)]$ of functions from the real interval $[0, 1]$ to $\mathbb{R}$ which satisfy the following conditions :

(i) $\overline{n}(r)$ is a bounded left continuous non-increasing function on $[0, 1]$,

(ii) $\underline{n}(r)$ is a bounded left continuous non-decreasing function on $[0, 1]$,

(iii) $\underline{n}(1) = \overline{n}(1) = n$.



Operations :

$$\widetilde{a} + \widetilde{b} = [\underline{a}(r) + \underline{b}(r),\ \overline{a}(r) + \overline{b}(r)],$$
$$-\widetilde{a} = [-\overline{a}(r),\ -\underline{a}(r)],$$

$\widetilde{a} = \widetilde{b}$ if and only if $\underline{a}(r) = \underline{b}(r)$ and $\overline{a}(r) = \overline{b}(r)$ for each real $r \in [0, 1]$

Passage from tuple to parametric

We consider a fuzzy number $\widetilde{n} = (n, \alpha, \beta)$ in the family L-R such that

$$\mu_{\widetilde{n}_-}(x) = L\left(\frac{n-x}{\gamma}\right), \quad \mu_{\widetilde{n}_+}(x) = R\left(\frac{x-n}{\delta}\right),$$

with $\alpha, \beta > 0$ and where L and R are bijectives.
For all $r \in [0,1]$, $\widetilde{n}_r = [\underline{n}(r), \overline{n}(r)]$ with

$$\underline{n}(r) = n - \alpha\, L^{-1}(r) \quad \text{et} \quad \overline{n}(r) = n + \beta\, R^{-1}(r)$$

The triangular case

$L = R = F$ where $F(x) = 1 - x$ is bijective with $F^{-1} = F$.

We get $\widetilde{n} = [\underline{n}, \overline{n}]$ with

$$\underline{n}(r) = \alpha\, r + n - \alpha \quad \text{and} \quad \overline{n}(r) = -\beta\, r + n + \beta \text{ for } r \in [0,1].$$

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Resolution Algorithm

- We start from the system of s polynomials in n variables :

$$F : \begin{cases} f_1(x_1, x_2, \ldots, x_n) = \widetilde{b_1} \\ \quad\vdots \\ f_s(x_1, x_2, \ldots, x_n) = \widetilde{b_s} \end{cases}$$

  where $x_1, x_2, \ldots, x_n$ are real variables and all the coefficients and values to the right of the equalities are triangular fuzzy numbers.

- We move to the parametric system P by replacing the fuzzy coefficients by their parametric representation

$$P : \begin{cases} f_{1,1}(x_1, x_2, \ldots, x_n, r) = \overline{b_1}(r) \\ f_{1,2}(x_1, x_2, \ldots, x_n, r) = \underline{b_1}(r) \\ \quad\vdots \\ f_{s,1}(x_1, x_2, \ldots, x_n, r) = \overline{b_s}(r) \\ f_{s,2}(x_1, x_2, \ldots, x_n, r) = \underline{b_s}(r) \end{cases}$$

  with $2s$ polynomials and $n+1$ variables $x_1, \ldots, x_n, r$ where $r \in [0, 1]$.
  All coefficients in F are triangular fuzzy numbers, so P is linear in r.

Resolution Algorithm

Therefore, the parametric system can be written as follows

$$P : \begin{cases} h_1(x_1, x_2, \ldots, x_n)r + g_1(x_1, x_2, \ldots, x_n) = 0 \\ h_2(x_1, x_2, \ldots, x_n)r + g_2(x_1, x_2, \ldots, x_n) = 0 \\ \quad \vdots \\ h_{2s}(x_1, x_2, \ldots, x_n)r + g_{2s}(x_1, x_2, \ldots, x_n) = 0 \end{cases}$$

where $h_i, g_i \in \mathbb{K}[x_1, x_2, \ldots, x_n]$.

- By collecting the coefficients h_i, g_i, we construct the collected crisp system F', satisfied for all $r \in [0, 1]$.

  The set of positive solutions of the starting system F is equal to the variety of the system of its collected crisp form.
- We compute a set of caracteristic sets Z for F' with the Wu Wen Tsun algorithm
- And we compute the variety of positive solutions V of F' i.e.

$$V(F') = \bigcup_{B \in Z} V(B/I_B)$$

  où $I_B = \prod_{b \in B} init(b)$

Example from an application in economics

Demand and supply are nonlinear polynomial functions of the price f_d and f_o, such that $q_d = f_d(p)$ and $q_o = f_o(p)$ with :

$$\begin{cases} q_d + a = b.p^2, \\ q_o + c = d.p^2 \end{cases}$$

where

- a, b, c et d : coefficients represented by triangular fuzzy numbers and q_o, q_d and p are real variables.

The variables are the quantity supplied q_o , the quantity demanded q_d and the price p.

The objective of the study is to achieve equality of supply and demand.
Let's put $q_d = q_o = x$ and $p = y$.

$$F : \begin{cases} x + (-1, 1, 1) = (-2, 1, 1)y^2, \\ x + (3, 1, 1) = (2, 1, 1)y^2 \end{cases}$$

This system is solved by the algorithm described above.

Example from an application in economics

- Computation of the parametric system of system F

$$\begin{cases} x + [r-2, -r] = [r-3, -r-1]y^2, \\ x + [r+2, -r+4] = [r+1, -r+3]y^2 \end{cases}$$
$$\Leftrightarrow \begin{cases} [x+r-2, x-r] = [y^2r - 3y^2, -y^2r - y^2], \\ [x+r+2, x-r+4] = [y^2r + y^2, -y^2r + 3y^2] \end{cases}$$
$$\Leftrightarrow \begin{cases} [(1-y^2)r + x + 3y^2 - 2, (y^2-1)r + x + y^2] = [0,0], \\ [(1-y^2)r + x - y^2 + 2, (y^2-1)r + x - 3y^2 + 4] = [0,0] \end{cases}$$

By identification, we obtain :

$$P : \begin{cases} (1-y^2)r + x + 3y^2 - 2 = 0, \\ (y^2-1)r + x + y^2 = 0, \\ (1-y^2)r + x - y^2 + 2 = 0, \\ (y^2-1)r + x - 3y^2 + 4 = 0 \end{cases}$$

- construction of the collected crisp system collected F'

$$F' : \begin{cases} (1-y^2) = 0, \\ (y^2-1) = 0, \\ x + 3y^2 - 2 = 0, \\ x + y^2 = 0, \\ x - y^2 + 2 = 0, \\ x - 3y^2 + 4 = 0 \end{cases}$$

- Wu's algorithm on the system F' returns the characteristic set $Z = [\{x+1, y^2-1\}]$
- We find the variety solution $V = \{(x=-1, y=\pm 1)\}$.
  Thus, all the solutions of F were obtained exactly by the method presented.

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Implementation of the Fuzzy Library

- Implementation of different representations in several classes
- Redefining operators on these objects (polymorphism), and methods for displaying graphs, order relations and transition to parametric representation

```
n1 = NombreFlou(1,2,2,1,"Lin","Quad")
n2 = NombreFlou(2,3,1,2,"Quad","Lin")
```

```
n3 = n1 - n2
print(res)
```

```
(-7,4,5,7), Lin - Quad
```

```
p1 = n1.forme_parametrique()
p2 = n2.forme_parametrique()
print(p1)
print(p2)
```

```
Symbolic Ring
[[2*sqrt(1/2)*sqrt(r) - 1, -2*sqrt(-1/2*r + 1/2) + 1],-r + 3] , Quad - Lin
[r + 1,[2*sqrt(-1/2*r + 1/2) + 3, -2*sqrt(1/2)*sqrt(r) + 5]] , Lin - Quad
```

```
p3 = p1 - p2
print(p3)
```

```
[[4*sqrt(1/2)*sqrt(r) - 6, -4*sqrt(-1/2*r + 1/2) - 2],-2*r + 2] , Quad - Lin
```

```
n3PM = n3.forme_parametrique()
print(n3PM)
```

```
Symbolic Ring
[[4*sqrt(1/2)*sqrt(r) - 6, -4*sqrt(-1/2*r + 1/2) - 2],-2*r + 2] , Quad - Lin
```

Implementation of the Fuzzy Library

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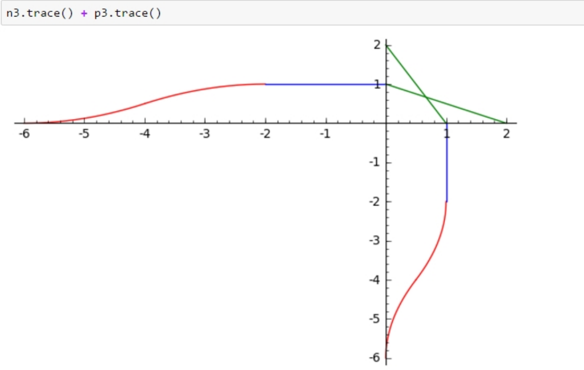


Implementation of the Fuzzy Library

➤ Resolution of polynomial systems with triangular fuzzy coefficients

```
A = PolynomialRing(QQ, 'x,y',order ='invlex')
B = A['r']

var = list(A.gens())
n = len(var)
x = var[0]
y = var[1]
x<y
```

```
True
```

```
dep=[(NombreFlouRed(1,0,0,"Lin","Lin")*x + NombreFlouRed(-1,1,1,"Lin","Lin"), NombreFlouRed(-2,1,1,"Lin","Lin")*y**2),\
     (NombreFlouRed(1,0,0,"Lin","Lin")*x + NombreFlouRed(3,1,1,"Lin","Lin"), NombreFlouRed(2,1,1,"Lin","Lin")*y**2)]
```

```
results = solveFuzzyPolynomialSystem(dep)
```

```
Le système tranché collecté est composé des polynômes suivants :
[-3*y^2 + x + 4, -y^2 + 1, -y^2 + x + 2, y^2 - 1, y^2 + x, 3*y^2 + x - 2]
Calcul de l'ensemble des ensembles caractéristiques du système tranché collecté :
D =
[[-3*y^2 + x + 4, -y^2 + 1, -y^2 + x + 2, y^2 - 1, y^2 + x, 3*y^2 + x - 2]]
Calcul de l'ensemble caractéristique :
S =
[-3*y^2 + x + 4, -y^2 + 1, -y^2 + x + 2, y^2 - 1, y^2 + x, 3*y^2 + x - 2]
Calcul d'un ensemble basique :
L'élément de rang minimal est -3*y^2 + x + 4
On ne garde dans l'ensemble que les éléments qui sont réduits par rapport à y^2 - 1/3*x - 4/3, on obtient l'ensemble []
Basic Set B =
[y^2 - 1/3*x - 4/3]
PREM(F,B) =
[0, x + 1, x + 1, x + 1, x + 1, x + 1]
S U PREM (F,B)\{0} =
[x + 1, y^2 - 1, -3*y^2 + x + 4, -y^2 + 1, y^2 + x, 3*y^2 + x - 2, -y^2 + x + 2]
Calcul d'un ensemble basique :
L'élément de rang minimal est x + 1
On ne garde dans l'ensemble que les éléments qui sont réduits par rapport à x + 1, on obtient l'ensemble [y^2 - 1, -y^2 + 1]
L'élément de rang minimal est y^2 - 1
On ne garde dans l'ensemble que les éléments qui sont réduits par rapport à y^2 - 1, on obtient l'ensemble []
Basic Set B =
[x + 1, y^2 - 1]
retour dans la fonction CharacSet
Charac Set =
[x + 1, y^2 - 1]
retour dans la fonction Wu
```

```
print(results)
```

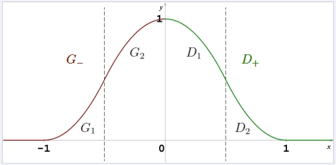
```
([[x + 1, y^2 - 1]], [1])
```

```
var('x'), var('y')
solve([x + 1 ==0, y^2 - 1==0],x,y)
```

```
[[x == -1, y == 1], [x == -1, y == -1]]
```

Exploration of the quadratic case

- G and D are decomposed to allow uniqueness of representation for $\widetilde{n}$ as in the triangular case

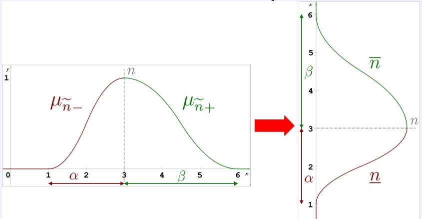


- In order to respect the uniqueness constraints of G and D for the quadratic family, constraints are applied to them, and we then have

$$G_1(x) = 2(x+1)^2,\ G_2(x) = 1 - 2x^2,\ D_1(x) = 1 - 2x^2,\ D_2(x) = 2(-x+1)^2.$$

We obtain $\widetilde{n} = [\underline{n}, \overline{n}]$ with

$$\underline{n}(r) = \begin{cases} b_1(r) = \sqrt{\frac{r}{2}} + n - \alpha & 0 \leq r \leq \frac{1}{2}, \\ b_2(r) = n - \alpha\sqrt{\frac{1-r}{2}} & \frac{1}{2} \leq r \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{et} \quad \overline{n}(r) = \begin{cases} h_1(r) = n + \beta\sqrt{\frac{1-r}{2}} & \frac{1}{2} \leq r \leq 1, \\ h_2(r) = -\beta\sqrt{\frac{r}{2}} + n + \beta & 0 \leq r \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$



Conclusion

Extension

Reinforce the resolution approach : new results which are independent from the spread functions L-R.

- The *real transform* $\mathcal{T}(S)$: formula that gives collected crisp system with only $3\,s$ equations of k variables with real coefficients
- We extended the result to other families of fuzzy numbers

Implementation

- Finding real solutions of polynomial systems through the management of the fuzzy system's solutions signs : computing positive solutions of 2^k systems.
- We propose an optimized algorithm called `SolveFuzzySystem` which reduces the number 2^k of systems to solve
- A parallel version of the algorithm is described

Thank you !